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Some new Farkas-type results for inequality systems with DC functions

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Abstract We present some Farkas-type results for inequality systems involving finitely many DC functions. To this end we use the so-called Fenchel-Lagrange duality approach applied to an optimization problem with DC objective function and DC inequality constraints. Some recently obtained Farkas-type results are rediscovered as special cases of our main result.

Keywords Farkas-type results · DC functions · Conjugate duality

1 Introduction

Since optimization techniques became more and more used in various fields of applications, an increasing number of problems that cannot be solved using the methods of linear or convex programming arised. Many of these problems are *DC optimization problems*, i.e. problems whose objective and/or constraint functions are functions which can be written as *differences of convex functions*. Although the largest number of the papers on this field present techniques of solving DC programming problems ([7,14,17,18]), the study of dual conditions characterizing global optimality has not been neglected ([2,5,6,9–13,16]).

The problem we treat in this paper consists in minimizing an extended real-valued DC function defined over the space $\mathbb{R}^n$ with respect to finitely many extended real-valued DC constraint functions. To this problem we determine its Fenchel-Lagrange-type dual problem,

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whose construction is described here in detail. The Fenchel-Lagrange dual problem is a "combination" of the well-known Fenchel and Lagrange duals and was recently introduced by Boţ and Wanka for convex optimization problems by means of the perturbation approach from the theory of conjugate duality ([1–4,21]). By using the same technique like Martínez-Legaz and Volle in [13], we reduce the study of the duality for DC problems to the study of the duality for convex optimization problems and in this way we define a dual problem for the primal DC one. A constraint qualification which guarantees the existence of strong duality is also given. Regarding other duality concepts for DC optimization problems we invite the reader to consult [11,12,19,20].

Recently in Refs. [3] and [4] some Farkas-type results for inequality systems involving finitely many convex constraints have been presented, by using an approach based on the theory of conjugate duality for convex optimization problems. The aim of this paper is to extend these results to inequality systems involving DC functions by using the duality theory developed for DC optimization problems. We shown that some results which can be found in the existing literature ([3,8,9]) arise as special cases of the problem we treat. More than that, we derive some equivalent formulations which rediscover some results given in the past in a general framework ([10,16]).

The paper is organized as follows. In Sect. 2 we present some definitions and results that are used later in the paper. In Sect. 3 we give a dual problem for the optimization problem with DC objective function and DC inequality constraints. Section 4 contains the main result of the paper; using the duality acquired in Sect. 3 we give a Farkas-type theorem for inequality systems involving DC function. In the last section we deal with some particular instances, rediscovering in this way some existing results in the literature.

2 Notations and preliminaries

In this section we introduce some notations and preliminary results which shall be used in the paper. All vectors are considered to be column vectors. Any column vector can be transposed to a row vector by an upper index ^{T}. By $x^T y = \sum_{i=1}^n x_i y_i$ we denote the usual inner product of two vectors $x = (x_1, \dots, x_n)^T$ and $y = (y_1, \dots, y_n)^T$ in the real space $\mathbb{R}^n$.

By $\operatorname{ri}(X)$, $\operatorname{co}(X)$ and $\operatorname{cl}(X)$ we will denote the *relative interior*, the *convex hull* and the *closure* of the set $X \subseteq \mathbb{R}^n$, respectively. Furthermore, the *cone* and the *convex cone* generated by the set X are denoted by $\operatorname{cone}(X) = \bigcup_{\lambda \geq 0} \lambda X$ and, respectively, $\operatorname{coneco}(X) = \bigcup_{\lambda \geq 0} \lambda \operatorname{co}(X)$. For an optimization problem (P) we denote by $v(P)$ its optimal objective value.

For a set $X \subseteq \mathbb{R}^n$ we consider the *indicator function of X*

$$\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}, \delta_X(x) = \begin{cases} 0, & x \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

and the *support function of X*, $\sigma_X : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\sigma_X(u) = \sup_{x \in X} u^T x$, respectively.

For a given function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we denote by $\operatorname{dom}(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$ its *effective domain* and by $\operatorname{epi}(f) = \{(x, r) : x \in \mathbb{R}^n, r \in \mathbb{R}, f(x) \leq r\}$ its *epigraph*, respectively. We say that f is *proper* if its effective domain is a nonempty set and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$.

When X is a nonempty subset of $\mathbb{R}^n$ we define for the function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ its *conjugate relative to the set X* by $f_X^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, $f_X^*(p) = \sup_{x \in X} \{p^T x - f(x)\}$. For $X = \mathbb{R}^n$ the conjugate relative to the set X is actually the *(Fenchel-Moreau) conjugate function* of f, $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, $f^*(p) = \sup_{x \in \mathbb{R}^n} \{p^T x - f(x)\}$.

For an arbitrary $x \in \mathbb{R}^n$ with $f(x) \in \mathbb{R}$ the *subdifferential* of the function f at x is the set

$$\partial f(x) = \{x^* \in \mathbb{R}^n : f(y) - f(x) \geq (y-x)^T x^*, \forall y \in \mathbb{R}^n\}.$$

The function f is said to be *subdifferentiable* at $x \in \mathbb{R}^n (f(x) \in \mathbb{R})$ if $\partial f(x) \neq \emptyset$. For all x and x^* in $\mathbb{R}^n$ we have $f(x) + f^*(x^*) \geq x^{*T} x$ (the *Young-Fenchel inequality*) and it can be shown that if $f(x) \in \mathbb{R}$

$$f(x) + f^*(x^*) = x^{*T} x \Leftrightarrow x^* \in \partial f(x). \tag{1}$$

Alongside the natural operations with $+\infty$ and $-\infty$ we adopt the following conventions (cf. [2,13])

$$(+\infty) - (+\infty) = (-\infty) - (-\infty) = (+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty,$$
$$0(+\infty) = +\infty \text{ and } 0(-\infty) = 0.$$

It is easy to see that for $f : \mathbb{R}^n \to \mathbb{R}$ the last two conventions imply $0f = \delta_{\text{dom}(f)}$.

Definition 2.1 Let the functions $f_1, \ldots, f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be given. The *infimal convolution* function of $f_1, \ldots, f_m$ is the function $f_1 \square \ldots \square f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$(f_1 \square \ldots \square f_m)(x) = \inf \left\{ \sum_{i=1}^m f_i(x_i) : x = \sum_{i=1}^m x_i \right\}.$$

Theorem 2.1 (cf. [15]) *Let $f_1, \ldots, f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \text{ri}(\text{dom}(f_i))$ is nonempty, then*

$$\left(\sum_{i=1}^m f_i\right)^*(p) = (f_1^* \square \ldots \square f_m^*)(p) = \inf \left\{ \sum_{i=1}^m f_i^*(p_i) : p = \sum_{i=1}^m p_i \right\}$$

and for each $p \in \mathbb{R}^n$ the infimum is attained.

A simple consequence of the Theorem 2.1 which closes this preliminary section follows.

Corollary 2.2 *Let $f_1, \ldots, f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \text{ri}(\text{dom}(f_i))$ is nonempty, then*

$$\text{epi}\left(\left(\sum_{i=1}^m f_i\right)^*\right) = \sum_{i=1}^m \text{epi}(f_i^*).$$

3 Duality for the DC programming problem

Let us consider the following optimization problem

$$(P) \qquad \inf_{\substack{x \in X, \\ g_i(x) - h_i(x) \leq 0, \\ i=1,\ldots,m}} \Big(g(x) - h(x)\Big),$$

where X is a nonempty convex subset of $\mathbb{R}^n$, $g, h : \mathbb{R}^n \to \overline{\mathbb{R}}$ are two proper convex functions and $g_i, h_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \ldots, m$, are proper convex functions such that

$$\bigcap_{i=1}^{m} \mathrm{ri}\left(\mathrm{dom}(g_i)\right) \bigcap \mathrm{ri}\left(\mathrm{dom}(g)\right) \bigcap \mathrm{ri}(X) \neq \emptyset. \tag{2}$$

We denote by $\mathcal{F}(P) = \left\{x \in X : g_i(x) - h_i(x) \leq 0, i = 1, \ldots, m\right\}$ the feasible set of (P) and we assume that $\mathcal{F}(P) \neq \emptyset$. Moreover, we assume that h is lower semicontinuous on $\mathcal{F}(P)$ and that h_i, $i = 1, \ldots, m$, are subdifferentiable on $\mathcal{F}(P)$.

Lemma 3.1 *It holds*

$$\mathcal{F}(P) = \bigcup_{\substack{y_i^* \in \mathrm{dom}(h_i^*),\\ i=1,\ldots,m}} \left\{x \in X : g_i(x) - {y_i^*}^T x + h_i^*(y_i^*) \leq 0, i = 1, \ldots, m\right\}.$$

Proof "$\subseteq$" Let $x \in \mathcal{F}(P)$. By the assumptions we made, it follows $x \in \cap_{i=1}^m \mathrm{dom}(h_i)$ and the existence of $y_i^* \in \partial h_i(x)$ for all $i = 1, \ldots, m$. By (1) we have $g_i(x) - {y_i^*}^T x + h_i^*(y_i^*) = g_i(x) - h_i(x) \leq 0$ for all $i = 1, \ldots, m$.

"$\supseteq$" For the opposite inclusion, let $y^* = (y_1^*, \ldots, y_m^*) \in \prod_{i=1}^{m} \mathrm{dom}\left(h_i^*\right)$ and $x \in X$ such that $g_i(x) - {y_i^*}^T x + h_i^*(y_i^*) \leq 0$ for all $i = 1, \ldots, m$. Thus, for $i = 1, \ldots, m$, $g_i(x) < +\infty$. Since (cf. the Young-Fenchel inequality) $g_i(x) - h_i(x) \leq g_i(x) - {y_i^*}^T x + h_i^*(y_i^*) \leq 0$, $i = 1, \ldots, m$, the conclusion follows. □

Throughout the paper we shall use the notation $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$ for $y_i^* \in \mathrm{dom}(h_i^*)$, $i = 1, \ldots, m$, where y^* is the m-tuple $(y_1^*, \ldots, y_m^*)$. By Lemma 3.1 we get an equivalent formulation for the optimal objective value of the problem (P).

Theorem 3.2 *Under the hypotheses considered in this section it holds*

$$v(P) = \inf_{\substack{x^* \in \mathrm{dom}(h^*),\\ y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)}} \inf_{\substack{x \in X,\\ g_i(x) - {y_i^*}^T x + h_i^*(y_i^*) \leq 0,\\ i=1,\ldots,m}} \left\{g(x) - {x^*}^T x + h^*(x^*)\right\}. \tag{3}$$

Proof Since h is proper, convex and lower semicontinuous on $\mathcal{F}(P)$ it holds

$$h(x) = h^{**}(x) = \sup_{x^* \in \mathrm{dom}(h^*)} \left\{{x^*}^T x - h^*(x^*)\right\}.$$

Thus,

$$v(P) = \inf_{x \in \mathcal{F}(P)} \Big(g(x) - h(x)\Big) = \inf_{x^* \in \mathrm{dom}(h^*)} \inf_{x \in \mathcal{F}(P)} \left\{g(x) - {x^*}^T x + h^*(x^*)\right\}.$$

Using the decomposition of the set $\mathcal{F}(P)$ given by Lemma 3.1, the conclusion is straightforward. □

Taking a careful look at relation (3), one may notice that the inner infimum can be seen as an optimization problem with a convex objective function and convex inequality constraints.

Thus it is quite natural to consider it as a separate optimization problem in order to deal with it by means of duality

$$(P_{x^*,y^*}) \qquad \inf_{\substack{x\in X,\\ g_i(x)-y_i^{*T}x+h_i^*(y_i^*)\leq 0,\\ i=1,\dots,m}} \left(g(x) - x^{*T}x + h^*(x^*)\right),$$

where $x^* \in \mathrm{dom}(h^*)$ and $y^* \in \prod_{i=1}^m \mathrm{dom}(h_i^*)$.

Let $x^* \in \mathrm{dom}(h^*)$ and $y^* \in \prod_{i=1}^m \mathrm{dom}(h_i^*)$ be fixed. We construct a dual problem for (P_{x^*,y^*}) and give sufficient conditions such that strong duality holds, i.e. the optimal objective value of the primal coincides with the optimal objective value of the dual and the dual has an optimal solution. Considering the functions $\widetilde{g} : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\widetilde{g}(x) = g(x) - x^{*T}x + h^*(x^*)$ and $\widetilde{g}_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\widetilde{g}_i(x) = g_i(x) - y_i^{*T}x + h_i^*(y_i^*)$, $i = 1, \dots, m$, the problem (P_{x^*,y^*}) can be equivalently written as

$$(P_{x^*,y^*}) \qquad \inf_{\substack{x\in X,\\ \widetilde{g}_i(x)\leq 0,\\ i=1,\dots,m}} \widetilde{g}(x).$$

One can notice that since g and g_i are proper and convex, the function $\widetilde{g}$ and $\widetilde{g}_i$ are proper and convex, too, for $i = 1, \dots, m$. Next we consider the Lagrange dual problem to (P_{x^*,y^*}) with $q = (q_1, \dots, q_m)^T \in \mathbb{R}_+^m$ as dual variable

$$(D_{x^*,y^*}) \qquad \sup_{q\geqq 0} \inf_{x\in X} \left\{\widetilde{g}(x) + \sum_{i=1}^m q_i \widetilde{g}_i(x)\right\}.$$

The inner infimum can be equivalently written as

$$\inf_{x\in X} \left\{\widetilde{g}(x) + \sum_{i=1}^m q_i \widetilde{g}_i(x)\right\} = -\left(\widetilde{g} + \sum_{i=1}^m q_i \widetilde{g}_i\right)_X^*(0).$$

Taking into consideration the convexity and the properness of the functions $\widetilde{g}$ and $\widetilde{g}_i$, $i = 1, \dots, m$, and that (2) is fulfiled, it follows by Theorem 2.1 that

$$\left(\widetilde{g} + \sum_{i=1}^m q_i \widetilde{g}_i\right)_X^*(0) = \left(\widetilde{g} + \sum_{i=1}^m \widetilde{g}_i + \delta_X\right)^*(0) =$$

$$\inf_{z\in\mathbb{R}^n} \left\{\widetilde{g}^*(z) + \left(\sum_{i=1}^m q_i \widetilde{g}_i + \delta_X\right)^*(-z)\right\} = \inf_{z\in\mathbb{R}^n} \left\{\widetilde{g}^*(z) + \left(\sum_{i=1}^m q_i \widetilde{g}_i\right)_X^*(-z)\right\}$$

and the infimum is attained for some $z \in \mathbb{R}^n$.

This leads to the following formulation for the dual (D_{x^*,y^*})

$$(D_{x^*,y^*}) \qquad \sup_{\substack{z\in\mathbb{R}^n,\\ q\geqq 0}} \left\{-\widetilde{g}^*(z) - \left(\sum_{i=1}^m q_i \widetilde{g}_i\right)_X^*(-z)\right\}.$$

Since $\widetilde{g}^*(z) = g^*(x^* + z) - h^*(x^*)$ and

$$\left(\sum_{i=1}^m q_i \widetilde{g}_i\right)_X^*(-z) = \left(\sum_{i=1}^m q_i g_i\right)_X^*\left(\sum_{i=1}^m q_i y_i^* - z\right) - \sum_{i=1}^m q_i h_i^*(y_i^*),$$

it follows immediately that the dual (D_{x^*,y^*}) has the form (we take $p := x^* + z$)

$$(D_{x^*,y^*}) \qquad \sup_{\substack{p\in\mathbb{R}^n,\\ q\geqq 0}} \left\{ h^*(x^*) + \sum_{i=1}^m q_i h_i^*(y_i^*) - g^*(p) - \left(\sum_{i=1}^m q_i g_i\right)_X^* \left(x^* + \sum_{i=1}^m q_i y_i^* - p\right) \right\}.$$

Theorem 3.3 *Between the primal problem (P_{x^*,y^*}) and the dual (D_{x^*,y^*}) weak duality is always satisfied, i.e. $v(P_{x^*,y^*}) \geq v(D_{x^*,y^*})$.*

Since in the general case strong duality can fail, in order to avoid such a situation we introduce the following generalized interior point constraint qualification (cf. [15])

$$(CQ_{y^*}) \left| \begin{array}{l} \exists x' \in \bigcap_{i=1}^m \operatorname{ri}\left(\operatorname{dom}(g_i)\right) \bigcap \operatorname{ri}\left(\operatorname{dom}(g)\right) \bigcap \operatorname{ri}(X) \text{ such that} \\ \left\{ \begin{array}{ll} g_i(x') - y_i^{*T} x' + h_i^*(y_i^*) \leq 0, & i \in L, \\ g_i(x') - x'^T y_i^* + h_i^*(y_i^*) < 0, & i \in N, \end{array} \right. \end{array} \right.$$

where $L := \left\{ i \in \{1, \ldots, m\} : g_i \text{ is an affine function} \right\}$ and $N := \{1, \ldots, m\} \backslash L$.

Regarding strong duality between (P_{x^*,y^*}) and (D_{x^*,y^*}) we have the following assertion.

Theorem 3.4 *Assume that $v(P_{x^*,y^*})$ is finite. If (CQ_{y^*}) is fulfiled, then between (P_{x^*,y^*}) and (D_{x^*,y^*}) strong duality holds, i.e. $v(P_{x^*,y^*}) = v(D_{x^*,y^*})$ and the dual problem has an optimal solution.*

Proof To the problem

$$(P_{x^*,y^*}) \qquad \inf_{\substack{x\in X,\\ \widetilde{g}_i(x)\leq 0,\\ i=1,\ldots,m}} \widetilde{g}(x)$$

we associate its Lagrange dual problem

$$\sup_{q\geqq 0} \inf_{x\in X} \left\{ \widetilde{g}(x) + \sum_{i=1}^m q_i \widetilde{g}_i(x) \right\}.$$

Since the condition (CQ_{y^*}) is fulfiled, by Theorem 28.2 in Ref. [15], the optimal objective values of (P_{x^*,y^*}) and its Lagrange dual are equal and, moreover, there exists an optimal solution $\overline{q} = (\overline{q}_1, \ldots, \overline{q}_m)^T \in \mathbb{R}_+^m$ of the dual such that

$$v(P_{x^*,y^*}) = \sup_{q\geqq 0} \inf_{x\in X} \left\{ \widetilde{g}(x) + \sum_{i=1}^m q_i \widetilde{g}_i(x) \right\} = \inf_{x\in X} \left\{ \widetilde{g}(x) + \sum_{i=1}^m \overline{q}_i \widetilde{g}_i(x) \right\}.$$

Further we deal with the infimum in the last term of the equality from above. As $\operatorname{dom}(\widetilde{g}) = \operatorname{dom}(g)$ and $\operatorname{dom}(\sum_{i=1}^m \overline{q}_i \widetilde{g}_i) = \cap_{i=1}^m \operatorname{dom}(\widetilde{g}_i) = \cap_{i=1}^m \operatorname{dom}(g_i)$, it holds

$$\operatorname{ri}\left(\operatorname{dom}(\widetilde{g})\right) \bigcap_{i=1}^m \operatorname{ri}\left(\operatorname{dom}\left(\sum_{i=1}^m \overline{q}_i \widetilde{g}_i\right)\right) \bigcap \operatorname{ri}\left(X\right) \neq \emptyset,$$

which implies that (cf. Theorem 2.1)

$$v(P_{x^*,y^*}) = \inf_{x\in X}\left\{\widetilde{g}(x) + \sum_{i=1}^m \overline{q}_i \widetilde{g}_i(x)\right\} = -\left(\widetilde{g} + \sum_{i=1}^m \overline{q}_i \widetilde{g}_i\right)_X^*(0)$$
$$= \sup_{z\in\mathbb{R}^n}\left\{ -\widetilde{g}^*(z) - \left(\sum_{i=1}^m \overline{q}_i \widetilde{g}_i\right)_X^*(-z)\right\}$$

and that there exists $\overline{z} \in \mathbb{R}^n$ such that the supremum is attained. Thus

$$v(P_{x^*,y^*}) = \sup_{z\in\mathbb{R}^n}\left\{ -\widetilde{g}^*(z) - \left(\sum_{i=1}^m \overline{q}_i \widetilde{g}_i\right)_X^*(-z)\right\} = -\widetilde{g}^*(\overline{z}) - \left(\sum_{i=1}^m \overline{q}_i \widetilde{g}_i\right)_X^*(-\overline{z})$$
$$= -g^*(x^* + \overline{z}) + h^*(x^*) - \left(\sum_{i=1}^m \overline{q}_i g_i\right)_X^*\left(\sum_{i=1}^m \overline{q}_i y_i^* - \overline{z}\right) + \sum_{i=1}^m \overline{q}_i h_i^*(y_i^*).$$

For $\overline{p} := x^* + \overline{z}$, it follows

$$v(P_{x^*,y^*}) = h^*(x^*) + \sum_{i=1}^m \overline{q}_i h_i^*(y_i^*) - g^*(\overline{p}) - \left(\sum_{i=1}^m \overline{q}_i g_i\right)_X^*\left(x^* + \sum_{i=1}^m \overline{q}_i y_i^* - \overline{p}\right)$$

and so we get that $v(P_{x^*,y^*}) = v(D_{x^*,y^*})$ and $(\overline{p}, \overline{q})$ is an optimal solution for (D_{x^*,y^*}). □

Taking into consideration the results given by Theorems 3.2 and 3.4, it seems natural to introduce the following dual problem to (P)

$$(D) \qquad \inf_{\substack{x^*\in\mathrm{dom}(h^*),\\ y^*\in\prod_{i=1}^m \mathrm{dom}(h_i^*)}} \ \sup_{\substack{q\geqq 0,\\ p\in\mathbb{R}^n}} \left\{h^*(x^*) + \sum_{i=1}^m q_i h_i^*(y_i^*) - g^*(p) - \left(\sum_{i=1}^m q_i g_i\right)_X^*\left(x^* + \sum_{i=1}^m q_i y_i^* - p\right)\right\}.$$

By the construction of (D) there is a weak duality statement for (P) and (D) as follows.

Theorem 3.5 *It holds* $v(P) \geq v(D)$.

Concerning the strong duality between (P) and (D), based on the the considerations done above we have the following theorem.

Theorem 3.6 *Let* (CQ_{y^*}) *be fulfiled for all* $y^* \in \prod_{i=1}^m \mathrm{dom}(h_i^*)$. *Then* $v(P) = v(D)$.

4 Farkas-type results for inequality systems with DC functions

By using of the duality theory developed above, we can give now the following Farkas-type result.

Theorem 4.1 *Suppose that* (CQ_{y^*}) *holds for all* $y^* \in \prod_{i=1}^m \mathrm{dom}(h_i^*)$. *Then the following assertions are equivalent:*

(i) $x \in X$, $g_i(x) - h_i(x) \leq 0$, $i = 1, \dots, m \Rightarrow g(x) - h(x) \geq 0$;

(ii) $\forall x^* \in \operatorname{dom}(h^*)$ *and* $\forall y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$, *there exist* $p \in \mathbb{R}^n$ *and* $q \geqq 0$ *such that*

$$h^*(x^*) + \sum_{i=1}^{m} q_i h_i^*(y_i^*) - g^*(p) - \left(\sum_{i=1}^{m} q_i g_i \right)_X^* \left(x^* + \sum_{i=1}^{m} q_i y_i^* - p \right) \geq 0. \quad (4)$$

Proof "$(i) \Rightarrow (ii)$" Let be $x^* \in \operatorname{dom}(h^*)$ and $y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$. The statement (i) implies $v(P) \geq 0$ and using Theorem 3.2 we acquire $v(P_{x^*,y^*}) \geq 0$. Since the assumptions of Theorem 3.4 are achieved, strong duality holds, i.e. $v(D_{x^*,y^*}) = v(P_{x^*,y^*}) \geq 0$ and the dual (D_{x^*,y^*}) has an optimal solution. Thus there exist $p \in \mathbb{R}^n$ and $q \geqq 0$ such that relation (4) is true.

"$(ii) \Rightarrow (i)$" Consider $x^* \in \operatorname{dom}(h^*)$ and $y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$. Then there exist $p \in \mathbb{R}^n$ and $q \geqq 0$ such that (4) is true and so

$$\sup_{\substack{p \in \mathbb{R}^n, \\ q \geqq 0}} \left\{ h^*(x^*) + \sum_{i=1}^{m} q_i h_i^*(y_i^*) - g^*(p) - \left(\sum_{i=1}^{m} q_i g_i \right)_X^* \left(x^* + \sum_{i=1}^{m} q_i y_i^* - p \right) \right\} \geq 0.$$

Since x^* and y^* were arbitrarily chosen we have $v(D) \geq 0$. Weak duality between (P) and (D) always holds and thus we obtain $v(P) \geq 0$, i.e. (i) is true. □

In the following we formulate Theorem 4.1 as a theorem of the alternative.

Corollary 4.2 *Assume the hypothesis of Theorem* 4.1 *fulfiled. Then either the inequality system*

$$(I) \qquad x \in X, g_i(x) - h_i(x) \leq 0, i = 1, \dots, m, g(x) - h(x) < 0$$

has a solution or each of the following systems

$$(II_{x^*,y^*})\, h^*(x^*) + \sum_{i=1}^{m} q_i h_i^*(y_i^*) - g^*(p) - \left(\sum_{i=1}^{m} q_i g_i \right)_X^* \left(x^* + \sum_{i=1}^{m} q_i y_i^* - p \right) \geq 0,$$
$$p \in \mathbb{R}^n, q \geqq 0,$$

where $x^* \in \operatorname{dom}(h^*)$ *and* $y_i^* \in \operatorname{dom}(h_i^*)$, $i = 1, \dots, m$, *has a solution, but never both.*

Next we give an equivalent assertion to the statement (ii) in Theorem 4.1 using the epigraphs of the functions involved.

Theorem 4.3 *The statement (ii) in Theorem* 4.1 *is equivalent to*

$$\operatorname{epi}(h^*) \subseteq \bigcap_{y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)} \left\{ \operatorname{epi}(g^*) + \operatorname{coneco}\left[\bigcup_{i=1}^{m} \left(\operatorname{epi}(g_i^*) - \left(y_i^*, h_i^*(y_i^*)\right) \right) \right] + \operatorname{epi}(\sigma_X) \right\}.$$

Proof "⇒" We prove that for an arbitrary $y^* = (y_1^*, \ldots, y_m^*)$ in the set $\prod_{i=1}^{m} \operatorname{dom}(h_i^*)$,

$$\operatorname{epi}(h^*) \subseteq \operatorname{epi}(g^*) + \operatorname{coneco}\left[\bigcup_{i=1}^{m}\left(\operatorname{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big)\right)\right] + \operatorname{epi}(\sigma_X). \tag{5}$$

To this end let (x^*, r) be a given element in $\operatorname{epi}(h^*)$. Thus $x^* \in \operatorname{dom}(h^*)$ and assertion (2) implies the existence of $p \in \mathbb{R}^n$ and $q \geqq 0$ such that the relation (4) is true. Consider first that $q = 0$. Relation (4) becomes $h^*(x^*) - g^*(p) - \delta_X^*(x^* - p) \geq 0$. Since $r \geq h^*(x^*)$ we have $r - g^*(p) \geq \delta_X^*(x^* - p)$. Thus $\big(x^*, r\big) = \big(p, g^*(p)\big) + \big(x^* - p, r - g^*(p)\big) \in \operatorname{epi}(g^*) + \operatorname{epi}(\sigma_X) \subseteq \operatorname{epi}(g^*) + \operatorname{coneco}\left[\bigcup_{i=1}^{m}\left(\operatorname{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big)\right)\right] + \operatorname{epi}(\sigma_X)$.

Now we assume that $q \neq 0$. The set $I_q = \{i : q_i \neq 0\}$ is obviously nonempty and relation (4) looks like

$$h^*(x^*) + \sum_{i\in I_q} q_i h_i^*(y_i^*) - g^*(p) - \left(\sum_{i\in I_q} q_i g_i\right)_X^*\left(x^* + \sum_{i\in I_q} q_i y_i^* - p\right) \geq 0.$$

In the hypotheses given by (2) we have

$$\left(\sum_{i\in I_q} q_i g_i\right)_X^*\left(x^* + \sum_{i\in I_q} q_i y_i^* - p\right) = \left(\sum_{i\in I_q} q_i g_i + \delta_X\right)^*\left(x^* + \sum_{i\in I_q} q_i y_i^* - p\right)$$
$$= \inf\left\{\sum_{i\in I_q}(q_i g_i)^*(v_i) + \sigma_X(z) : x^* + \sum_{i\in I_q} q_i y_i^* - p = \sum_{i\in I_q} v_i + z\right\},$$

and this infimum is attained for some vectors z and v_i, $i \in I_q$, in $\mathbb{R}^n$. It follows that

$$h^*(x^*) + \sum_{i\in I_q} q_i h_i^*(y_i^*) \geq g^*(p) + \sum_{i\in I_q}(q_i g_i)^*(v_i) + \sigma_X(z),$$

where $x^* + \sum_{i\in I_q} q_i y_i^* - p = \sum_{i\in I_q} v_i + z$. Since $q_i > 0$, $i \in I_q$, we have $(q_i g_i)^*(v_i) = q_i g_i^*\left(\frac{1}{q_i} v_i\right)$. Considering the vectors $v_i' \in \mathbb{R}^n$, $v_i' := \frac{1}{q_i} v_i$, $i \in I_q$, we get $x^* = p + \sum_{i\in I_q} q_i\big(v_i' - y_i^*\big) + z$ and

$$r \geq h^*(x^*) \geq g^*(p) + \sum_{i\in I_q} q_i\left(g_i^*\big(v_i'\big) - h_i^*(y_i^*)\right) + \sigma_X(z).$$

On the other hand, because of

$$\left(v_i' - y_i^*, g_i^*\big(v_i'\big) - h_i^*\big(y_i^*\big)\right) \in \bigcup_{i=1}^{m}\left(\operatorname{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big)\right), i \in I_q,$$

we have

$$\left(\sum_{i\in I_q} q_i\big(v_i' - y_i^*\big), \sum_{i\in I_q} q_i\big(g_i^*\big(v_i'\big) - h_i^*(y_i^*)\big)\right) = \sum_{i\in I_q} q_i\left(v_i' - y_i^*, g_i^*(v_i') - h_i^*(y_i^*)\right)$$
$$\in \operatorname{coneco}\left[\bigcup_{i=1}^{m}\left(\operatorname{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big)\right)\right],$$

which implies that

$$\big(x^*, r\big) \in \operatorname{epi}(g^*) + \operatorname{coneco}\left[\bigcup_{i=1}^{m}\left(\operatorname{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big)\right)\right] + \operatorname{epi}(\sigma_X).$$

"⇐" Let us consider $x^* \in \operatorname{dom}(h^*)$ and $y^* \in \prod_{i=1}^{m} \operatorname{dom}(h_i^*)$. Since

$$\begin{aligned}&\big(x^*, h^*(x^*)\big) \in \operatorname{epi}(h^*) \subseteq \\ &\operatorname{epi}(g^*) + \operatorname{coneco}\left[\bigcup_{i=1}^{m}\left(\operatorname{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big)\right)\right] + \operatorname{epi}(\sigma_X),\end{aligned}$$

there exist $(p, r) \in \operatorname{epi}(g^*)$, $(v, s) \in \operatorname{coneco}\left[\bigcup_{i=1}^{m}\left(\operatorname{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big)\right)\right]$ and $(z, t) \in \operatorname{epi}(\sigma_X)$ such that

$$\big(x^*, h^*(x^*)\big) = (p, r) + (v, s) + (z, t). \tag{6}$$

Moreover, there exist $\lambda \geq 0$, $\mu_i \geq 0$ and $(v_i, s_i) \in \operatorname{epi}(g_i^*) - \big(y_i^*, h_i^*(y_i^*)\big)$, $i = 1, \ldots, m$, such that $\sum_{i=1}^{m} \mu_i = 1$ and

$$(v, s) = \lambda \sum_{i=1}^{m} \mu_i (v_i, s_i). \tag{7}$$

For all $i \in \{1, \ldots, m\}$ we have $\big(v_i + y_i^*, s_i + h_i^*(y_i^*)\big) \in \operatorname{epi}(g_i^*)$ and it follows immediately that

$$g_i^*(v_i + y_i^*) - h_i^*(y_i^*) \leq s_i. \tag{8}$$

Assume first that $\lambda = 0$. Then $(v, s) = (0, 0)$ and relation (6) becomes $\big(x^*, h^*(x^*)\big) = (p, r) + (z, t)$. Since $r \geq g^*(p)$ and $t \geq \sigma_X(z) = \delta_X^*(z)$, this implies $x^* = p + z$ and $h^*(x^*) \geq g^*(p) + \delta_X^*(z)$. Considering $q := (0, \ldots, 0)^T \in \mathbb{R}^m$ it holds

$$h^*(x^*) \geq g^*(p) + \left(\sum_{i=1}^{m} q_i g_i + \delta_X\right)^* \left(x^* + \sum_{i=1}^{m} q_i v_i - p\right),$$

and the conclusion is straightforward.

In case $\lambda > 0$, let us consider the vector $q := (\lambda\mu_1, \ldots, \lambda\mu_m)^T \in \mathbb{R}^m$. Since it holds $\sum_{i=1}^{m} \mu_i = 1$, the set I_q is obviously nonempty and relation (7) becomes $(v, s) = \sum_{i \in I_q} q_i (v_i, s_i)$. Taking into consideration relation (8) we obtain $v = \sum_{i \in I_q} q_i v_i$ and $s = \sum_{i \in I_q} q_i s_i \geq \sum_{i \in I_q} q_i \big(g_i^*(v_i + y_i^*) - h_i^*(y_i^*)\big)$. Combining these two results with relation (6) and with the inequalities $g^*(p) \leq r$ and $\delta_X^*(z) = \sigma_X(z) \leq t$ we obtain $x^* = p + \sum_{i \in I_q} q_i v_i + z$ and

$$h^*(x^*) \geq g^*(p) + \sum_{i \in I_q} q_i \big(g_i^*(v_i + y_i^*) - h_i^*(y_i^*)\big) + \delta_X^*(z).$$

By the properties of the conjugate of the sum of a family of functions we obtain

$$\sum_{i\in I_q} q_i g_i^*(v_i + y_i^*) + \delta_X^*(z) = \sum_{i\in I_q} \left(q_i g_i\right)^* \left(q_i v_i + q_i y_i^*\right) + \delta_X^*(z)$$
$$\geq \left(\sum_{i\in I_q} q_i g_i + \delta_X\right)^* \left(\sum_{i\in I_q} q_i y_i^* + \sum_{i\in I_q} q_i v_i + z\right)$$
$$= \left(\sum_{i\in I_q} q_i g_i\right)_X^* \left(\sum_{i\in I_q} q_i y_i^* + x^* - p\right) = \left(\sum_{i=1}^m q_i g_i\right)_X^* \left(x^* + \sum_{i=1}^m q_i y_i^* - p\right).$$

The desired conclusion arises easily. □

Remark 4.1 Dual geometrical characterizations for the solvability of inequality systems expressed by means of inclusion relations involving the epigraphs of the conjugates of the functions involved have been given in the past in Refs. [10,16] (even in very general settings), [8] and [3]. The last theorem shows that the geometrical characterization for the solvability of inequality systems from Ref. [10] can be expressed by using the objective function of the Fenchel-Lagrange-type dual of the DC primal problem. On the other hand, one can see that in the finite dimensional setting the regularity conditions given in Ref. [10] can be replaced by generalized Slater-type constraint qualifications.

5 Special cases

In this section we give Farkas-type results for some problems which turn out to be special cases of the problem (P).

5.1 The case $h = 0$

The problem (P) becomes in this case an optimization problem with a convex objective function and finitely many DC constraint functions.

Since in this case $\text{dom}(h^*) = \{0\}$ and $\text{epi}(h^*) = \{0\} \times [0, +\infty)$, the following theorems can be easily obtained from Theorems 4.1 and 4.3, respectively.

Theorem 5.1 *Suppose that* (CQ_{y^*}) *holds for all* $y^* \in \prod_{i=1}^m \text{dom}(h_i^*)$. *Then the following assertions are equivalent:*

(i) $x \in X,\ g_i(x) - h_i(x) \leq 0,\ i = 1, \ldots, m \Rightarrow g(x) \geq 0$;
(ii) $\forall y^* \in \prod_{i=1}^m \text{dom}(h_i^*)$, *there exist* $p \in \mathbb{R}^n$ *and* $q \geqq 0$ *such that*

$$\sum_{i=1}^m q_i h_i^*(y_i^*) - g^*(p) - \left(\sum_{i=1}^m q_i g_i\right)_X^* \left(\sum_{i=1}^m q_i y_i^* - p\right) \geq 0.$$

Theorem 5.2 *The statement (ii) in Theorem* 5.1 *is equivalent to*

$$0 \in \bigcap_{y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)} \left\{ \mathrm{epi}(g^*) + \mathrm{coneco}\left[\bigcup_{i=1}^{m} \left(\mathrm{epi}(g_i^*) - \left(y_i^*, h_i^*(y_i^*)\right)\right)\right] + \mathrm{epi}(\sigma_X) \right\}. \tag{9}$$

Proof Theorem 4.3 ensures the equivalence between the statement (ii) in Theorem 5.1 and the relation

$$\{0\} \times [0, +\infty) \subseteq \bigcap_{y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)} \left\{ \mathrm{epi}(g^*) + \mathrm{coneco}\left[\bigcup_{i=1}^{m} \left(\mathrm{epi}(g_i^*) - \left(y_i^*, h_i^*(y_i^*)\right)\right)\right] + \mathrm{epi}(\sigma_X) \right\}.$$

By using the definition of the epigraph of a function one can see that this is nothing else than (9). □

Remark 5.1 In this section we rediscover the Farkas-type results given in Ref. [3].

5.2 The case $h_i = 0, i = 1, \ldots, m$

The problem (P) is now an optimization problem with a DC objective function and finitely many convex constraint functions.

It is obvious that for all $i = 1, \ldots, m$ we have

$$h_i^*(y_i^*) = \begin{cases} +\infty, & y_i^* \neq 0, \\ 0, & y_i^* = 0. \end{cases}$$

Thus, we have $\prod_{i=1}^{m} \mathrm{dom}(h_i^*) = \{(0, \ldots, 0)\}$ and the constraint qualification (CQ_{y^*}) for $y^* \in \prod_{i=1}^{m} \mathrm{dom}(h_i^*)$ turns out to be

$$(CQ_0)\ \exists x' \in \mathrm{ri}(X) \bigcap \mathrm{ri}(\mathrm{dom}(g)) \bigcap_{i=1}^{m} \mathrm{ri}\left(\mathrm{dom}(g_i)\right) : \begin{cases} g_i(x') \leq 0, & i \in L, \\ g_i(x') < 0, & i \in N. \end{cases}$$

Theorem 5.3 *Suppose that* (CQ_0) *holds. Then the following assertions are equivalent:*

(i) $x \in X,\ g_i(x) \leq 0,\ i = 1, \ldots, m \Rightarrow g(x) - h(x) \geq 0$*;*
(ii) $\forall x^* \in \mathrm{dom}(h^*)$*, there exist* $p \in \mathbb{R}^n$ *and* $q \geqq 0$ *such that*

$$h^*(x^*) - g^*(p) - \left(\sum_{i=1}^{m} q_i g_i\right)_X^* (x^* - p) \geq 0.$$

Theorem 5.4 *The statement (ii) in Theorem 5.2 is equivalent to*

$$\mathrm{epi}(h^*) \subseteq \mathrm{epi}(g^*) + \mathrm{coneco}\left[\bigcup_{i=1}^{m} \mathrm{epi}(g_i^*)\right] + \mathrm{epi}(\sigma_X).$$

Both Theorems 5.3 and 5.4 are again direct consequences of Theorems 4.1 and 4.3, respectively. They express, as particular cases of our general result in Sect. 4, the outcomes obtained by Boţ and Wanka in Ref. [3] and by Jeyakumar and Glover in Ref. [9].

Remark 5.2 A very helpful characterization for the existent of a ε-optimal solutions for the optimization problem with DC objective function and convex inequality constraints has been given in Ref. [10] by means of the ε-subdifferentials of the functions involved.

5.3 The case $h = 0$ and $h_i = 0, i = 1, \ldots, m$

In this case our initial problem turns out to be a standard convex optimization problem with a convex objective function and finitely many convex constraint functions. The constraint qualification becomes also (CQ_0).

This special case has been treated in [3], [8] in finite dimensional spaces, but also in [16] and [10] in infinite dimensional spaces. Let us mention that our results are identical to the ones in [3], where alongside convex inequalities the inequality systems contain also some geometrical constraints.

Theorem 5.5 *Suppose that* (CQ_0) *holds. Then the following assertions are equivalent:*

(i) $x \in X, g_i(x) \leq 0, i = 1, \ldots, m \Rightarrow g(x) \geq 0$*;*
(ii) there exist $p \in \mathbb{R}^n$ *and* $q \geqq 0$ *such that*

$$g^*(p) + \left(\sum_{i=1}^{m} q_i g_i\right)_X^*(-p) \leq 0.$$

Theorem 5.6 *The statement (ii) in Theorem* 5.5 *is equivalent to*

$$0 \in \mathrm{epi}(g^*) + \mathrm{coneco}\left[\bigcup_{i=1}^{m} \mathrm{epi}(g_i^*)\right] + \mathrm{epi}(\sigma_X).$$

6 Conclusions

In this paper we present Farkas-type result for inequality systems with finitely many DC functions. The approach we use is based on the conjugate duality for an optimization problem with DC objective function and DC inequality constraints. We generalize and rediscover some results given in the past in the literature. Also the connection which exist between the Farkas-type results and the theory of the alternative and, respectively, the theory of duality is put in a new light and underlined once more.

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